Stichting
MATHEMATISCH CENTRUM
2e Boerhaavestraat 49
Amsterdam

R 183

Calculations Concerning a Filter

1952

Calculations concerning a filter.

The following calculations were performed on behalf of the N.S.F., O.C.L.T., Hilversum and according to data given by Mr. Irani.

The work consisted of two main parts: calculations of some integrals and a filter calculation.

I. The integrals were given as:

$$I(\omega)= 2\,K_1\,\omega_1 \int_0^{\omega} \frac{d\omega}{(\omega^2-\omega_1^2)\sqrt{(1-\omega^2)(1-k^2\omega^2)}} -2\bar{K}_1\bar{\omega}_1 \int_0^{\omega} \frac{d\omega}{(\omega^2-\bar{\omega}_1^2)\sqrt{(1-\omega^2)(1-k^2\omega^2)}}$$

$$+ 2K_2\omega_2 \int_0^{\omega} \frac{d\omega}{(\omega^2-\omega_2^2)\sqrt{(1-\omega^2)(1-k^2\omega^2)}} -2\bar{K}_2\bar{\omega}_2 \int_0^{\omega} \frac{d\omega}{(\omega^2-\bar{\omega}_2^2)\sqrt{(1-\omega^2)(1-k^2\omega^2)}}$$

$$+ i\ C\ sn^{-1}\omega\ . \qquad (1)$$

where $K_1 = \sqrt{(1-\omega_1^2)(1-k^2\omega_1^2)}$ and $\omega_1 = 0{,}1876 + 0{,}6269\ i$

$K_2 = \sqrt{(1-\omega_2^2)(1-k^2\omega_2^2)}$ $\qquad \omega_2 = 0{,}5751 + 0{,}4553\ i$

$k = 0{,}8$

C had to be chosen so that $I(1)= 7\pi i$.

The calculation of the integrals can be performed in two ways:

1. by putting $\omega = sn\ u$

$$2K_1\omega_1 \int_0^{\omega} \frac{d\omega}{(\omega^2-\omega_1^2)\sqrt{(1-\omega^2)(1-k^2\omega^2)}} = -2K_1\omega_1^{-1} \int_0^{sn^{-1}\omega} \frac{d\omega}{1-\frac{1}{\omega_1^2}\,sn^2 u} =$$

$$= -2K_1\omega_1^{-1}\left[\ sn^{-1}\omega + \frac{1}{\omega_1^2}\int_0^{sn^{-1}\omega} \frac{sn^2 u\ du}{1-\frac{1}{\omega_1^2}\,sn^2 u}\right] =$$

$$= -2\ K_1\omega_1^{-1}\ sn^{-1}\omega\ +2\ \Pi\ (sn^{-1}\omega_1, a)=$$

$$= -2\ K_1\omega_1^{-1} sn^{-1}\omega\ + 2\ sn^{-1}\omega\ .\ Z(a)+ \log\frac{\Theta(u-a)}{\Theta(u+a)} \qquad (2)$$

(see L.M.Milne-Thomson: Jacobian Elliptic Function Tables)

where a is given by $\frac{1}{\omega_1^2} = k^2\ sn^2\ a$ $\qquad (3)$

For $I(\omega)$ we get: $I(\omega)= \log \frac{\Theta(u-a)}{\Theta(u+a)} \cdot \frac{\Theta(u+\bar{a})}{\Theta(u-\bar{a})} \cdot \frac{\Theta(u-c)}{\Theta(u+c)} \cdot \frac{\Theta(u+\bar{c})}{\Theta(u-\bar{c})} +$

$$\operatorname{sn}^{-1}\omega \left\{ -2K_1\omega_1^{-1}+2\bar{K}_1\bar{\omega}_1^{-1}-2K_2\omega_2^{-1}+2\bar{K}_2\bar{\omega}_2^{-1}+2Z(a)-2Z(\bar{a})+2Z(c)-2Z(\bar{c}) + iC\right\} \quad (4)$$

where c is defined by $\frac{1}{\omega_2^2} = k^2 \operatorname{sn}^2 c$. (3a)

Due to the periodicity of $\Theta(u)$, the logarithm is zero for $\omega = 1$. C was computed from

$$7\pi i= \operatorname{sn}^{-1} 1 \left\{-2K_1\omega_1^{-1}+2\bar{K}_1\bar{\omega}_1^{-1}-2K_2\omega_2^{-1}+2\bar{K}_2\bar{\omega}_2^{-1}+2Z(a)-2Z(\bar{a})+ +2Z(c)-2Z(\bar{c})+ iC\right\}.$$

Next $Z(\omega)$ was computed as function of $u = \operatorname{sn}^{-1}\omega$, and the points u where $I(\omega)$ equals $2\pi i$, $4\pi i$, $6\pi i$ were interpolated. Afterwards the values of ω in these points were found.

2. The second way of finding these ω-values is by computing the integrals in (1) numerically. This can easily be done after the substitution $v = \sqrt{1-\omega}$. The coefficient C was obtained from the condition $I(1)= 7\pi i$, and the values of ω where I equals $2\pi i$, $4\pi i$, $6\pi i$ were obtained by interpolating in the table made of $I(\omega)$ as function of ω.

The answers of both ways were the same: $C = 3.2258$

$F(\omega_3)= 2\pi i$	$\omega_3= 0.3850$
$F(\omega_4)= 4\pi i$	$\omega_4= 0.7342$
$F(\omega_5)= 6\pi i$	$\omega_5= 0.9692$.

From $\omega = 1$ to $\omega = \frac{1}{k}$ the imaginary part of $I(\omega)$ remains constant and the real part increases from 0 to 11.79.

This was found by numerical integration. It was checked by doing it twice. It was possible to check it in a similar manner as above, but the transcendental equations (3) became much more complicated.

From $\omega = 1.25$ to $\omega = \infty$ the real part of $I(\omega)$ remains constant but the imaginary part decreases from $7\pi i$ to $4\pi i$. Now the point ω_6 had to be computed where $\operatorname{Im} F(\omega) = 6\pi i$. This again was done by numerical integration. $\omega_6 = 1.3223$.

To check our computations the integrals were computed from $\omega = \frac{1}{k}$ to $\omega = \infty$ and the imaginary part was $-3\pi i$, so $\operatorname{Im} I(\infty) = 4\pi i$ as it should be.

II. With the values $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$ two polynomials h^2 and f^2 were constructed as follows:

$$h^2 = \omega^2(\omega^2-\omega_3^2)^2 \; (\omega^2-\omega_4^2)^2 \; (\omega^2-\omega_5^2)^2.$$

$$f^2 = (\omega^2-\omega_6^2)^2 \; (\omega^2-\omega_1^2)(\omega^2-\bar{\omega}_1^2)(\omega^2-\omega_2^2)(\omega^2-\bar{\omega}_2^2).$$

Furthermore $F(\omega)$ was constructed:

$F(\omega) = 10 \log_{10}\left\{ n \frac{h^2}{f^2} + 1 \right\}$; n was computed from the condition

$F(1/k) = 30.$

$n = 238.83.$

$F(\omega)$ was computed and is given in table I.

Next $\frac{g^2}{f^2}$ was defined by:

$$\frac{g^2}{f^2} = n \frac{h^2}{f^2} + 1.$$

Now the roots of the polynomial g^2 had to be found. They were found by the method of dividing out real quadratic forms (See Milne: Numerical Analysis, p. 53).

The roots are:

	$\pm$ 0,21908 i
$\pm$ 0,39082	$\pm$ 0,20743 i
$\pm$ 0,74758	$\pm$ 0,18584 i
$\pm$ 1,00962	$\pm$ 0,06969 i

Next $\frac{g}{f}$ was defined as follows:

g has those zeros of g^2, that lay in the upper halfplane;

f has those zeros of f^2, that lay in the lower halfplane plus single zeros at ω_6 and $-\omega_6$.

Now arg g/f was computed. This is given in table II.

TABLE I

ω^2	$F(\omega)$	ω^2	$F(\omega)$
0	0	1.1	4.3399
0.05	0.4995	1.2	9.9090
0.1	0.1624	1.3	15.3362
0.15	0.0015	1.4	20.5807
0.2	0.1397	1.5	26.0808
0.25	0.3695	1.6	32.7452
0.3	0.5050	1.7	44.3785
0.35	0.4895	1.8	45.5385
0.4	0.3439	1.9	37.6832
0.45	0.1741	2.0	34.6510
0.5	0.0368	2.1	32.9933
0.55	0.0030	2.2	31.9673
0.6	0.0839	2.3	31.2917
0.65	0.2387	2.4	30.8319
0.7	0.4001	2.5	30.5138
0.75	0.4976	2.6	30.2934
0.8	0.4798	2.7	30.1432
0.85	0.3290	2.8	30.0438
0.9	0.1017	2.9	29.9827
0.95	0.0113	3.0	29.9507
1.0	0.5003	3.1	29.9411
		3.2	29.9488
		4.0	30.3239
		5.0	31.0106

Extrema of $F(\omega)$:

ω^2	$F(\omega)$
0.04	0.5254
0.32	0.5165
0.77	0.5061
3 1	29.94

TABLE II

ω	arg g/f	
0	-90°	68
0.1	-22°	65
0.2	43°	65
0.3	108°	70
0.4	178°	68
0.5	246°	65
0.6	311°	65
0.7	376°	67
0.8	443°	62
0.9	505°	84
1.0	589°	